# COVARIANT FIRST ORDER DIFFERENTIAL CALCULUS ON QUANTUM PROJECTIVE SPACES

MARTIN WELK

Abstract. We investigate covariant first order differential calculi on the quantum complex projective spaces $\mathrm{CP}_q^{N-1}$ which are quantum homogeneous spaces for the quantum group $\mathrm{SU}_q(N)$. Hereby, one more well-studied example of covariant first order differential calculus on a quantum homogeneous space is given. Since the complex projective spaces are subalgebras of the quantum spheres $\mathrm{S}_q^{2N-1}$ introduced by Vaksman and Soibelman, we get also an example of the relations between covariant differential calculus on two closely related quantum spaces.

Two approaches are combined in obtaining covariant first order differential calculi on $\mathrm{CP}_q^{N-1}$: 1. restriction of covariant first order differential calculi from $\mathrm{S}_q^{2N-1}$; 2. classification of calculi under appropriate constraints, using methods from representation theory.

The main result is that under three reasonable settings of dimension constraints, covariant first order differential calculi on $\mathrm{CP}_q^{N-1}$ exist and are (for $N \geq 6$) uniquely determined. This is a clear difference as compared to the case of the quantum spheres where several parametrical series of calculi exist. For two of the constraint settings, the covariant first order calculi on $\mathrm{CP}_q^{N-1}$ are also obtained by restriction from calculi on $\mathrm{S}_q^{2N-1}$ as well as from calculi on the quantum group $\mathrm{SU}_q(N)$.

## 1. Introduction

During the last decade, covariant differential calculus on quantum groups has been under intensive investigation. One main reason for this interest is that quantum groups are important examples of noncommutative geometric spaces, equipped with a rich additional structure. The description of differential calculus on them is an indispensable prerequisite for any analysis of their geometric structure.

Fundamental concepts of covariant differential calculus on quantum groups have been introduced in the work of Woronowicz [13]. Covariant first order differential calculi on quantum groups have been constructed; bicovariant first order differential calculi on the most important quantum groups have been classified [7], [8]. Higher order differential calculus has been studied [9], and basic concepts of differential geometry on quantum groups have been established.

Quantum spaces for quantum groups, and in particular quantum homogeneous spaces, are a wider class of noncommutative geometric spaces which still have a rich algebraic structure that can hoped to be helpful in investigating their geometric properties; they are in many aspects close to quantum groups. However, unlike for quantum groups, one is still far from having a comprehensive view on differential calculus on quantum homogeneous spaces. Only a small

number of examples have been studied in detail—apart from quantum vector spaces which have a comparatively simple structure, we should mention firstly Podleś' quantum spheres for which a classification of covariant first order differential calculi was given in [1]. First order differential calculi on the quantum spheres $\mathrm{S}_q^{2N-1}$ as introduced by Vaksman and Soibelman [10] has been classified by the author in a previous paper [11], [12]. The construction of these calculi by restricting covariant calculi from the quantum group $\mathrm{SU}_q(N)$ has been investigated in [6].

In this paper, we want to study the quantum projective spaces $\mathrm{CP}_q^{N-1}$ which are, as $\mathrm{S}_q^{2N-1}$, quantum homogeneous spaces for the quantum group $\mathrm{SU}_q(N)$, and which are in close relation to the quantum spheres. With this investigation, one further well-understood example of covariant first order differential calculus on a quantum homogeneous space is provided; moreover, the close relationship between $\mathrm{CP}_q^{N-1}$ and $\mathrm{S}_q^{2N-1}$ allows to ask how this relationship is reflected in the differential calculus. It is hoped that this work will contribute to a deeper understanding of covariant differential calculus on quantum homogeneous spaces since the study of a number of particular examples with different properties forms the ground on which more general constructions for differential calculi on quantum spaces and further theoretical work could be based.

## 2. The Quantum Projective Spaces $\mathrm{CP}_q^{N-1}$

To start with, we need to recall some basic definitions on quantum spaces. Our terminology essentially follows [4], [1].

Suppose $\mathcal{A}$ is a Hopf algebra with comultiplication $\Delta$ and counit $\varepsilon$. A pair $(X, \Delta_{\mathrm{R}})$ consisting of a unital algebra $X$ and an algebra homomorphism $\Delta_{\mathrm{R}} : X \to X \otimes \mathcal{A}$ is called a *quantum space for* $\mathcal{A}$ if $(\Delta_{\mathrm{R}} \otimes \mathrm{id})\Delta_{\mathrm{R}} = (\mathrm{id} \otimes \Delta)\Delta_{\mathrm{R}}$ and $(\mathrm{id} \otimes \varepsilon)\Delta_{\mathrm{R}} = \mathrm{id}$. Then, $\Delta_{\mathrm{R}}$ is called *(right) coaction* of $\mathcal{A}$ on $X$. A quantum space $(X, \Delta_{\mathrm{R}})$ (or simply, $X$) for $\mathcal{A}$ is a *quantum homogeneous space for* $\mathcal{A}$ if there exists an embedding $\iota : X \to \mathcal{A}$ with $\Delta_{\mathrm{R}} = \Delta \circ \iota$, i. e. $X$ can be considered as a sub-algebra of $\mathcal{A}$ with the coaction being the restricted comultiplication.

For our example, the deformation parameter $q$ will always be a real number, $q \neq 0, \pm 1$. Further, $N$ is a natural number, $N \geq 2$, parametrising the dimension of the underlying quantum group $\mathrm{SU}_q(N)$. Throughout the following, the R-matrix $\hat{R}$ which describes the commutation relations of the quantum group $\mathrm{SU}_q(N)$ [4], [5], [7] will play an important role. This is an invertible $N^2 \times N^2$-matrix with $\hat{R}^- := \hat{R} - (q - q^{-1})I$ as its inverse, where $I$ is the $N^2 \times N^2$ unit matrix. The entries of $\hat{R}$ are given by

$$\hat{R}^{ij}_{kl} = \begin{cases} 1 & \text{for } i = l \neq k = j, \\ q & \text{for } i = j = k = l, \\ q - q^{-1} & \text{for } i = k < j = l, \\ 0 & \text{otherwise.} \end{cases}$$

For abbreviation, we shall also use the following matrices which are derived from these fundamental ones:

$$\check{R}^{ij}_{kl} := \hat{R}^{lk}_{ji}; \qquad \grave{R}^{ij}_{kl} := q^{2l-2i}\hat{R}^{jl}_{ik}; \qquad \acute{R}^{ij}_{kl} := \hat{R}^{ki}_{lj};$$

$$\check{R}^{-ij}_{\ \ kl} := \hat{R}^{-lk}_{\ \ ji}; \qquad \grave{R}^{-ij}_{\ \ kl} := q^{2l-2i}\hat{R}^{-jl}_{\ \ ik}; \qquad \acute{R}^{-ij}_{\ \ kl} := \hat{R}^{-ki}_{\ \ lj};$$

$$\hat{\underline{R}}^{stuv}_{ijkl} := \grave{R}^{-tu}_{\ \ ab}\hat{R}^{sa}_{ic}\acute{R}^{cb}_{jk}\delta_{vl}; \qquad \hat{\underline{R}}^{-stuv}_{\ \ ijkl} := \grave{R}^{-tu}_{\ \ ab}\hat{R}^{-sa}_{\ \ ic}\acute{R}^{cb}_{jk}\delta_{vl};$$
$$\check{\underline{R}}^{stuv}_{ijkl} := \grave{R}^{-tu}_{\ \ ab}\check{R}^{bv}_{cl}\acute{R}^{ac}_{jk}\delta_{si}; \qquad \check{\underline{R}}^{-stuv}_{\ \ ijkl} := \grave{R}^{-tu}_{\ \ ab}\check{R}^{-bv}_{\ \ cl}\acute{R}^{ac}_{jk}\delta_{si}.$$

Finally, we define

$$\mathfrak{s}_+ := \sum_{i=0}^{N-1} q^{2i}; \quad \mathfrak{s}'_+ := \mathfrak{s}_+ - 1; \quad \mathfrak{s}''_+ := \mathfrak{s}'_+ - q^2; \quad \mathfrak{s}'''_+ := \mathfrak{s}''_+ - q^4; \quad \mathfrak{s}^{\mathsf{IV}}_+ := \mathfrak{s}'''_+ - q^6.$$

We can now define the quantum projective spaces to be considered here. These are also given in [4, 11.6]. Let $\mathrm{CP}_q^{N-1}$ be the algebra with $N^2$ generators $x_{ij}$, $1 \le i,j \le N$, and relations

$$\hat{\underline{R}}^{-stuv}_{\ \ ijkl} x_{st}x_{uv} = q^{-1}x_{ij}x_{kl}, \qquad \check{\underline{R}}^{stuv}_{ijkl} x_{st}x_{uv} = q x_{ij}x_{kl}, \qquad \sum_{i=1}^{N} x_{ii} = 1.$$

The further relation $\sum\limits_{j=1}^{N} q^{-2j}x_{ij}x_{jk} = x_{ik}$ is implied by these ones. The algebra $\mathrm{CP}_q^{N-1}$ can be equipped with a $*$-structure by letting $(x_{ij})^* := x_{ji}$. By the embedding $\iota : \mathrm{CP}_q^{N-1} \to \mathrm{SU}_q(N)$, $x_{ij} \mapsto u^1_i(u^1_j)^* = u^1_i S(u^j_1)$ where $u^i_j$ are the $N^2$ coordinates, and $S$ the antipode map of $\mathrm{SU}_q(N)$, $\mathrm{CP}_q^{N-1}$ becomes a quantum homogeneous space for $\mathrm{SU}_q(N)$, with the coaction $\Delta_{\mathrm{R}}(x_{ij}) = \sum\limits_{k,l} x_{kl} \otimes u^k_i S(u^j_l)$. We shall call $\mathrm{CP}_q^{N-1}$ *quantum projective space.*

The quantum space $\mathrm{CP}_q^{N-1}$ can be embedded as a quantum homogeneous space—i.e. respecting its algebra structure and $\mathrm{SU}_q(N)$-coalgebra structure—into the quantum sphere $\mathrm{S}_q^{2N-1}$ which has been studied in [5], [10], [4], [11]. The embedding is given by $x_{ij} \mapsto z_i z_j^*$ where $z_i$, $z_i^*$, $1 \le i \le N$, are the algebra generators of $\mathrm{S}_q^{2N-1}$.

Note that in the literature there exist two different versions of quantum projective space. The one used here belongs to a one-parameter series of quantum projective spaces $\mathcal{B}_q^\sigma$ described in [4, 11.6].

## 3. First order differential calculus

The following definitions concerning differential calculus are again in concordance with, [4], [1]. By a first order differential calculus on an algebra $X$ we shall mean a pair $(\Gamma, \mathrm{d})$ of a bimodule $\Gamma$ over $X$ and a linear mapping $\mathrm{d} : X \to \Gamma$ fulfilling Leibniz' rule $\mathrm{d}(xy) = (\mathrm{d}x)y + x(\mathrm{d}y)$ for all $x, y \in X$, and $\Gamma = \mathrm{Lin}\{x\mathrm{d}y \mid x, y \in X\}$. The elements of $\Gamma$ are called one-forms.

We call a first order differential calculus $(\Gamma, \mathrm{d})$ on a quantum space $(X, \Delta_{\mathrm{R}})$ for $\mathcal{A}$ (right) covariant if there exists a linear mapping $\Phi_{\mathrm{R}} : \Gamma \to \Gamma \otimes \mathcal{A}$ which satisfies the identities $(\Phi_{\mathrm{R}} \otimes \mathrm{id})\Phi_{\mathrm{R}} = (\mathrm{id} \otimes \Delta)\Phi_{\mathrm{R}}$; $(\mathrm{id} \otimes \varepsilon)\Phi_{\mathrm{R}} = \mathrm{id}$; $\Phi_{\mathrm{R}}(x\omega y) = \Delta_{\mathrm{R}}(x)\Phi_{\mathrm{R}}(\omega)\Delta_{\mathrm{R}}(y)$; $\Phi_{\mathrm{R}}(\mathrm{d}x) = (\mathrm{d} \otimes \mathrm{id})\Delta_{\mathrm{R}}(x)$ for all $x, y \in X$, $\omega \in \Gamma$. Those one-forms $\omega$ for which the identity $\Delta_{\mathrm{R}}(\omega) = \omega \otimes 1$ holds are called invariant. Remember that in the description of bicovariant differential calculi on quantum

groups, invariant one-forms play a central role since essentially all one-forms can be described by using only left- (or right-) invariant forms. On quantum homogeneous spaces there exist usually not enough invariant one-forms to enable a similar description; nevertheless, they still are of great significance for the differential calculus.

We mention that if $X$ is a $*$-algebra then the notion of a $*$-calculus can be introduced: $(\Gamma, \mathrm{d})$ is a $*$-calculus if $\sum_k x_k \mathrm{d}y_k = 0$ for $x_k, y_k \in X$ always implies $\sum_k \mathrm{d}(y_k^*) x_k^* = 0$. However, in this paper $*$-calculi won't play an important role.

From the definition of a covariant differential calculus and from the relations of the algebra $\mathrm{CP}_q^{N-1}$ it is clear that in any covariant first order differential calculus on $\mathrm{CP}_q^{N-1}$, the one-form $\Omega = \sum_{i,j=1}^{N} q^{-2j} x_{ij} \mathrm{d}x_{ji}$ is invariant (it might, however, be zero).

## 4. Covariant differential calculi on $\mathrm{CP}_q^{N-1}$

In this paper, two approaches will be used to obtain covariant first order differential calculi on the quantum projective spaces.

The first way is based on restricting differential calculi on $\mathrm{S}_q^{2N-1}$ to the subalgebra $\mathrm{CP}_q^{N-1}$. Differential calculi on $\mathrm{S}_q^{2N-1}$ have been classified in [11] under suitable settings for the classification constraints. In order to find out which of the calculi listed there can be restricted to $\mathrm{CP}_q^{N-1}$, one tries to calculate the bimodule structures of restricted calculi. To this purpose, expressions of the type $\mathrm{d}x \cdot y$ with $x, y \in \mathrm{CP}_q^{N-1}$ need to be transformed into left-module expressions using the bimodule structure of a $\mathrm{S}_q^{2N-1}$-calculus. The crucial question then is how to recognise which expressions on the right-hand side can be written in terms of $\mathrm{CP}_q^{N-1}$ only. Moreover, the left-module relations which may hold in the module of one-forms over $\mathrm{CP}_q^{N-1}$ have to be described. Up to this point, we have no effective algorithm to solve these two problems.

The second approach consists in direct classification of covariant first order differential calculi on $\mathrm{CP}_q^{N-1}$ under appropriate algebraic constraints, using representation theory similarly as done for Podleś' quantum spheres in [1] or for the Vaksman-Soibelman quantum spheres in [11]. However, choosing appropriate constraints turns out more difficult than in the case of, e. g., the quantum spheres $\mathrm{S}_q^{2N-1}$. One frequently used constraint setting requires the differentials of the algebra generators to generate the bimodule of one-forms as a free left module. Unlike for many other examples, this standard setting is obviously inadequate here from a geometrical point of view since the dimension of the differential calculus would then be much higher than that of the algebra itself. Although we are going to consider this setting, we shall look for more appropriate constraints. These should at one hand be some kind of a natural choice while on the other hand they should reduce the dimension of the differential calculus such that it becomes close to that of the underlying quantum space.

Fortunately, a combination of the two approaches makes it much easier to overcome the difficulties in both of them. The consideration of co-representations

which is the first step in following the classification strategy, leads to a precise knowledge about the types of expressions that may occur in the bimodule structure of covariant first order differential calculi on $\mathrm{CP}_q^{N-1}$. Thus, it is much easier to decide whether restricting a given $\mathrm{S}_q^{2N-1}$ calculus leads in fact to a differential calculus on the sub-algebra. On the other hand, if some differential calculi on $\mathrm{CP}_q^{N-1}$ can be obtained by restricting calculi from the quantum spheres, the relations which hold in these calculi will give evidence which type of algebraic relations should be introduced into the classification constraint.

Nevertheless, the proof of the classification results still involves rather complicated calculations which require the aid of computer algebra.

4.1. **First result: restriction of calculi from the quantum spheres.** In [11] differential calculi on the quantum spheres $\mathrm{S}_q^{2N-1}$ were classified. Two different classification constraints were applied. (The classification is complete for $N \geq 4$ but all calculi described exist for $N \geq 2$, too.) We recall the main results: There are four families $\Gamma_{\alpha\tau}$, $\Gamma'_{\alpha\omega}$, $\Gamma''_{\omega\psi}$, $\Gamma'''_{\varrho\tau}$ of covariant first order differential $*$-calculi with $\{\mathrm{d}z_1, \dots, \mathrm{d}z_N, \mathrm{d}z_1^*, \dots, \mathrm{d}z_N^*\}$ as a free left module basis for the bimodule of one-forms. Each has two real parameters, with the exception of certain parameter pairs for $\Gamma'_{\alpha\omega}$ with non-real $\alpha$ which we shall not include in our consideration. Further, there are three families $\tilde\Gamma_\lambda$, $\tilde\Gamma'_\lambda$, $\tilde\Gamma''_\lambda$ of covariant first order differential $*$-calculi, each with one real parameter, for which the bimodule of one-forms is generated as a left module by $\{\mathrm{d}z_1, \dots, \mathrm{d}z_N, \mathrm{d}z_1^*, \dots, \mathrm{d}z_N^*\}$ and for which all algebraic relations in the left module of one-forms are generated by one relation $\sum\limits_{i=1}^{N} z_i \mathrm{d}z_i^* + \lambda \sum\limits_{i=1}^{N} q^{-2i} z_i^* \mathrm{d}z_i = 0$. (We restrict ourselves to $*$-calculi here, leaving aside $\tilde\Gamma^{\bullet}_\lambda$ and $\tilde\Gamma^{\bullet\bullet}_\lambda$.) The equations taken verbatim from [11] which characterise the bimodule structure of these calculi, are given in section 6, equations (2)–(10). We can now state our first result.

**Theorem 1.** *Each of the covariant first order differential $*$-calculi $\Gamma_{\alpha\tau}$, $\Gamma'_{\alpha\omega}$, $\Gamma''_{\omega\psi}$, $\Gamma'''_{\varrho\tau}$, $\tilde\Gamma_\lambda$, $\tilde\Gamma'_\lambda$, $\tilde\Gamma''_\lambda$ on $\mathrm{S}_q^{2N-1}$ can be restricted to a covariant first order differential calculus on $\mathrm{CP}_q^{N-1}$. All of the calculi $\Gamma_{\alpha\tau}$, $\Gamma'''_{\varrho\tau}$ and $\tilde\Gamma''_\lambda$ yield the same restricted calculus $\underline{\tilde\Gamma}$ while all of the calculi $\Gamma'_{\alpha\omega}$, $\Gamma''_{\omega\psi}$, $\tilde\Gamma_\lambda$ and $\tilde\Gamma'_\lambda$ lead to the same restricted calculus $\underline{\tilde{\tilde\Gamma}}$, independent on the values of all parameters involved.*

*In $\underline{\tilde\Gamma}$, all relations in the left module of one-forms are generated by the set of relations ($1 \leq i, j \leq N$)*

$$\mathrm{d}x_{ij} = q^2 \sum_{s=1}^{N} q^{-2s} x_{is} \mathrm{d}x_{sj} + q^{-1} \sum_{a=1}^{N} \grave{R}^{-tu}_{\ bc} \hat{R}^{-sb}_{\ ia} \check{R}^{cv}_{aj} x_{st} \mathrm{d}x_{uv} - \frac{\mathfrak{s}''_+}{\mathfrak{s}'_+} x_{ij} \underline{\Omega}$$
$$- \frac{1}{\mathfrak{s}'_+} \delta_{ij} q^{2j} \underline{\Omega}.$$

*The bimodule structure of $\underline{\tilde\Gamma}$ is given by*

$$\mathrm{d}x_{ij} x_{kl} = q^{-2} \sum_{a=1}^{N} \hat{\underline{R}}^{-xyzw}_{\ ijkl} \grave{R}^{-tu}_{\ bc} \hat{R}^{-sb}_{\ za} \check{R}^{cv}_{aw} x_{xy} x_{st} \mathrm{d}x_{uv}$$

$$+ q^3 \check{\underline{R}}^{stuv}_{ijkl} \sum_{w=1}^{N} q^{-2w} x_{st} x_{uw} \mathrm{d}x_{wv} - \frac{q^{2N+2}}{\mathfrak{s}'_+} \delta_{jk} x_{ij} \underline{\Omega}$$
$$- \frac{q^{-1}}{\mathfrak{s}'_+} \acute{R}^{ab}_{jk} \hat{R}^{-}{}^{sc}_{ia} \check{R}^{cv}_{bl} x_{sv} \underline{\Omega} - \frac{q^{-2}}{\mathfrak{s}'_+} \delta_{ij} q^{2j} x_{kl} \underline{\Omega} - \frac{q^{-2} \mathfrak{s}^{\mathsf{IV}}_+}{\mathfrak{s}'_+} x_{ij} x_{kl} \underline{\Omega}.$$

*In $\tilde{\tilde{\underline{\Gamma}}}$, all relations in the left modules of one-forms are generated by the set of relations $(1 \le i, j \le N)$*

$$\mathrm{d}x_{ij} = q^2 \sum_{s=1}^{N} q^{-2s} x_{is} \mathrm{d}x_{sj} + q^{-1} \sum_{a=1}^{N} \grave{R}^{-}{}^{tu}_{bc} \hat{R}^{-}{}^{sb}_{ia} \check{R}^{cv}_{aj} x_{st} \mathrm{d}x_{uv}; \qquad \underline{\Omega} = 0.$$

*The bimodule structure of $\tilde{\tilde{\underline{\Gamma}}}$ is given by*

$$\mathrm{d}x_{ij} x_{kl} = q^{-2} \sum_{a=1}^{N} \hat{\underline{R}}^{-}{}^{xyzw}_{ijkl} \grave{R}^{-}{}^{tu}_{bc} \hat{R}^{-}{}^{sb}_{za} \check{R}^{cv}_{aw} x_{xy} x_{st} \mathrm{d}x_{uv}$$
$$+ q^3 \check{\underline{R}}^{stuv}_{ijkl} \sum_{w=1}^{N} q^{-2w} x_{st} x_{uw} \mathrm{d}x_{wv}.$$

*Remarks:* 1. Note that the first group of relations and the bimodule structure of $\tilde{\tilde{\underline{\Gamma}}}$ are obtained from the relations and bimodule structure of $\tilde{\underline{\Gamma}}$ just by inserting the additional relation $\underline{\Omega} = 0$. Thus, the calculus $\tilde{\tilde{\underline{\Gamma}}}$ is obtained by factorising $\tilde{\underline{\Gamma}}$ by this relation.
2. It was proved in [6] that the calculi $\tilde{\Gamma}_\lambda$ and $\tilde{\Gamma}''_{q^{2N+2}}$ on $\mathrm{S}_q^{2N-1}$ are restrictions of covariant (as for $\tilde{\Gamma}''_{q^{2N+2}}$, even bicovariant) first order differential calculi on $\mathrm{SU}_q(N)$. Since restriction of $\tilde{\Gamma}_\lambda$ to $\mathrm{CP}_q^{N-1}$ yields $\tilde{\tilde{\underline{\Gamma}}}$, while $\tilde{\Gamma}''_{q^{2N+2}}$ can be restricted to $\tilde{\underline{\Gamma}}$, both $\tilde{\underline{\Gamma}}$ and $\tilde{\tilde{\underline{\Gamma}}}$ are even restrictions of covariant differential calculi on $\mathrm{SU}_q(N)$.

4.2. **Second result: direct classification.** For a classification of covariant first order differential calculi on the quantum projective space $\mathrm{CP}_q^{N-1}$, we need a plausible constraint setting which should essentially consist of a dimension restriction for the bimodule of one-forms. We shall consider three settings for the classification constraint. First, we require that the bimodule of one-forms be generated by $\mathrm{d}x_{ij}$, $1 \le i, j \le N$, $(i, j) \ne (N, N)$ as free left module basis. As stated above, this constraint does not make much sense from the geometrical point of view since it means that the module of one-forms needs to be of far higher dimension than the underlying algebra itself. We consider this setting mostly for algebraic completeness since this type of condition is the starting-point for all other types of dimension condition taken into consideration.

The other two settings—which are supposed to be of geometrical relevance—are motivated by Theorem 1. We suppose that the differential calculi obtained by restriction of calculi from $\mathrm{S}_q^{2N-1}$ have appropriate dimension. Therefore, we choose the relations found in the calculi $\tilde{\underline{\Gamma}}$ and $\tilde{\tilde{\underline{\Gamma}}}$ as "templates" for our second

and third constraint setting; the actual constraints are obtained by allowing the coefficients in the relations to vary.

It turns out that allowing the coefficients to vary is in fact no essential generalisation because the values taken by the coefficients in the relations of the restricted calculi $\tilde{\underline{\Gamma}}$ and $\tilde{\tilde{\underline{\Gamma}}}$ are the only possible ones; finally, covariant differential calculi exist and are uniquely determined under all three constraints. The following theorem states our classification results for all these settings.

**Theorem 2.**

(i) *There is a covariant first order differential calculus* $(\Gamma, \mathrm{d}) = (\underline{\Gamma}, \mathrm{d})$ *on* $\mathrm{CP}_q^{N-1}$ *for which* $\{\mathrm{d}x_{ij} \mid i,j = 1,\ldots,N; (i,j) \neq (N,N)\}$ *is a free left module basis of* $\Gamma$. *If* $N \geq 6$, *then* $(\underline{\Gamma}, \mathrm{d})$ *is the only differential calculus with this property. The bimodule structure of* $(\underline{\Gamma}, \mathrm{d})$ *is given by*

$$\begin{aligned}
\mathrm{d}x_{ij} \cdot x_{kl} = {} & q^{-1} \underline{\hat{R}}^{-\,stuv}_{\;\;ijkl} x_{st} \mathrm{d}x_{uv} + q \underline{\check{R}}^{stuv}_{ijkl} x_{st} \mathrm{d}x_{uv} + \underline{\hat{R}}^{-\,stuv}_{\;\;wxyz} \underline{\check{R}}^{wxyz}_{ijkl} x_{st} \mathrm{d}x_{uv} \\
& - \sum_{w=1}^{N} q^{-2w} x_{ij} x_{kw} \mathrm{d}x_{wl} - \sum_{w=1}^{N} q^{-2w} \underline{\check{R}}^{stuv}_{ijkl} x_{st} x_{uw} \mathrm{d}x_{wv} \\
& - q \sum_{a=1}^{N} \grave{R}^{-\,tu}_{\;\;bc} \hat{R}^{-\,sb}_{\;\;ka} \check{R}^{cv}_{al} x_{ij} x_{st} \mathrm{d}x_{uv} \\
& - \sum_{a=1}^{N} \underline{\hat{R}}^{-\,xyzw}_{\;\;ijkl} \grave{R}^{-\,tu}_{\;\;bc} \hat{R}^{-\,sb}_{\;\;za} \check{R}^{cv}_{aw} x_{xy} x_{st} \mathrm{d}x_{uv} + (q^2+1) x_{ij} x_{kl} \underline{\Omega}
\end{aligned}$$

(ii) *For* $N \geq 6$, *there is exactly one covariant first order differential calculus* $(\Gamma, \mathrm{d})$ *on* $\mathrm{CP}_q^{N-1}$ *for which* $\{\mathrm{d}x_{ij} \mid i,j = 1,\ldots,N; (i,j) \neq (N,N)\}$ *generates* $\Gamma$ *as a left module, and for which all relations in the left module* $\Gamma$ *are algebraically generated by the set of relations* $(1 \leq i,j \leq N)$

$$\begin{aligned}
\mathrm{d}x_{ij} = {} & A \sum_{s=1}^{N} q^{-2s} x_{is} \mathrm{d}x_{sj} + B \sum_{a=1}^{N} \grave{R}^{-\,tu}_{\;\;bc} \hat{R}^{-\,sb}_{\;\;ia} \check{R}^{cv}_{aj} x_{st} \mathrm{d}x_{uv} \\
& + C x_{ij} \underline{\Omega} + D \delta_{ij} q^{2j} \underline{\Omega}
\end{aligned}$$

*for some fixed coefficients* $A$, $B$, $C$, *and* $D$. *This is the calculus* $(\Gamma, \mathrm{d}) = (\tilde{\underline{\Gamma}}, \mathrm{d})$ *from Theorem 1 with*

$$A = q^2; \qquad B = q^{-1}; \qquad C = -\frac{\mathfrak{s}''_+}{\mathfrak{s}'_+}; \qquad D = -\frac{1}{\mathfrak{s}'_+}.$$

(iii) *For* $N \geq 6$, *there is exactly one covariant first order differential calculus* $(\Gamma, \mathrm{d})$ *on* $\mathrm{CP}_q^{N-1}$ *for which* $\{\mathrm{d}x_{ij} \mid i,j = 1,\ldots,N; (i,j) \neq (N,N)\}$ *generates* $\Gamma$ *as a left module, and for which all relations in the left module* $\Gamma$ *are algebraically generated by the set of relations* $(1 \leq i,j \leq N)$

$$\mathrm{d}x_{ij} = A \sum_{s=1}^{N} q^{-2s} x_{is} \mathrm{d}x_{sj} + B \sum_{a=1}^{N} \grave{R}^{-\,tu}_{\;\;bc} \hat{R}^{-\,sb}_{\;\;ia} \check{R}^{cv}_{aj} x_{st} \mathrm{d}x_{uv}; \qquad \underline{\Omega} = 0$$

*for some fixed coefficients $A$, $B$. This is the calculus* $(\Gamma, \mathrm{d}) = (\tilde{\tilde{\underline{\Gamma}}}, \mathrm{d})$ *from Theorem 1 with*

$$A = q^2; \qquad\qquad B = q^{-1}.$$

*Remarks:* 1. It is easily seen that, like $\tilde{\tilde{\underline{\Gamma}}}$ from $\tilde{\tilde{\underline{\Gamma}}}$, even $\tilde{\underline{\Gamma}}$ (and thus, $\tilde{\tilde{\underline{\Gamma}}}$) is obtained from $\underline{\Gamma}$ by factorisation because the bimodule structure of $\underline{\Gamma}$ turns into that of $\tilde{\underline{\Gamma}}$ if simply the set of left-module relations of $\tilde{\underline{\Gamma}}$ is imposed.
2. All differential calculi discussed exist for $N \geq 2$. For the second and third case this is part of the statement of Theorem 1, so it had to be stated explicitly here only for case (i). However, the uniqueness is guaranteed only for $N \geq 6$; the reasons will become clear from the proof (see sections 5 and 7).

## 5. Representation theory

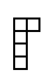
We start by investigating corepresentations of the quantum group $\mathrm{SU}_q(N)$ on $\mathrm{CP}_q^{N-1}$. By the coaction $\Delta_{\mathrm{R}}$, a corepresentation of $\mathrm{SU}_q(N)$ on $\mathrm{CP}_q^{N-1}$ is given which decomposes into summands corresponding to invariant vector spaces $V(k)$, $k = 0, 1, \ldots$ Here, $V(k)$ is the vector space of homogeneous polynomials which is generated by precisely those monomials of degree $k$ in the generators $x_{ij}$ which are not reduced to lower degree by the algebra relations of $\mathrm{CP}_q^{N-1}$. We denote by $\pi(k)$ the corepresentation of $\mathrm{SU}_q(N)$ on $V(k)$.

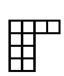
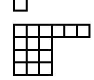
Since $q$ is not a root of unity, the representation theory is essentially identical to the classical case—see [3]—and the decomposition of corepresentations into irreducible summands can be described by means of Young frames, cf. [2]. We shall use this notation in the following, denoting the trivial corepresentation by $\mathbf{(0)}$.

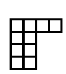
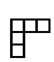
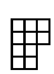
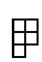
The bimodule structure of any covariant first order differential calculus needs to be formed by intertwining morphisms $T \in \mathrm{Mor}((\pi(1) + \pi(0)) \otimes (\pi(1) + \pi(0)), \pi(k) \otimes (\pi(1) + \pi(0)))$. (Note that the vector space generated by $x_{ij}$, $1 \leq i, j \leq N$, is $V(0) \oplus V(1)$.)

We show the calculations with Young frames for $N = 5$. For other $N \geq 4$, the calculations are analogous, while for $N = 2$, $N = 3$ additional coincidences of Young frames and, thus, irreducible summands have to be observed, spoiling the uniqueness argument in these cases. We shall have to sharpen the requirement even to $N \geq 6$ because of a linear independence argument used later.

Starting from $\pi(0) = \mathbf{(0)}$ and $\pi(1) =$ , one calculates successively $\pi(k)$ and $\pi(k) \otimes (\pi(1) + \pi(0))$, $k = 0, 1, \ldots$; note that $\pi(k+1)$ is obtained from $\pi(k)$ by cancelling all those summands from $\pi(k) \otimes (\pi(1) + \pi(0))$ which correspond to invariant subspaces annihilated by the algebra relations of $\mathrm{CP}_q^{N-1}$.

$\pi(0) = \mathbf{(0)}$, $\qquad$ $\pi(1) =$ ,

$\pi(2) =$ , $\qquad$ $\pi(3) =$ , $\ldots$;

$(\pi(1) + \pi(0)) \otimes (\pi(1) + \pi(0)) =$  $+$  $+$  $+$  $+ 4$  $+ 2\mathbf{(0)}$;

$$
\begin{aligned}
\pi(0)\otimes(\pi(1)+\pi(0)) &= \ydiagram{2,1,1,1} + \mathbf{(0)},\\
\pi(1)\otimes(\pi(1)+\pi(0)) &= \ydiagram{4,2,2,2} + \ydiagram{3,1,1} + \ydiagram{3,3,2,2} + \ydiagram{2,2,1} + 3\,\ydiagram{2,1,1,1} + \mathbf{(0)},\\
\pi(2)\otimes(\pi(1)+\pi(0)) &= \ydiagram{6,3,3,3} + \ydiagram{5,2,2,1} + \ydiagram{5,4,3,3} + \ydiagram{4,3,2,1} + 3\,\ydiagram{4,2,2,2}\\
&\quad + \ydiagram{3,1,1} + \ydiagram{3,3,2,2} + \ydiagram{2,1,1,1},\\
\pi(3)\otimes(\pi(1)+\pi(0)) &= \ydiagram{8,4,4,4} + \ydiagram{7,3,3,2} + \ydiagram{7,5,4,4} + \ydiagram{6,4,3,2} + 2\,\ydiagram{6,3,3,3}\\
&\quad + \ydiagram{5,2,2,1} + \ydiagram{5,4,3,3} + \ydiagram{4,2,2,2}
\end{aligned}
$$

By comparing these decompositions, it is found that $\pi(k)\otimes(\pi(1)+\pi(0))$ has two summands in common with $(\pi(1)+\pi(0))\otimes(\pi(1)+\pi(0))$ for $k=0$, five for $k=1$, four for $k=2$, one for $k=3$, and none for higher $k$. Taking into account the multiplicities of all these summands—e.g. $\ydiagram{2,1,1,1}$ occurs with multiplicity 4 in $(\pi(1)+\pi(0))\otimes(\pi(1)+\pi(0))$, and 5 in $\sum_k \pi(k)\otimes(\pi(1)+\pi(0))$, allowing for 20 independent subspace mappings—it can be seen that up to 33 morphisms can occur. Because of the necessary condition $\sum_{i=1}^{N} \mathrm{d}x_{ii} = 0$ some of the summands vanish automatically, and we are left with a general ansatz for the bimodule structure of a covariant differential calculus on $\mathrm{CP}_q^{N-1}$ containing 27 morphisms, namely

$$
\begin{aligned}
\mathrm{d}x_{ij}x_{kl} &= a_1 x_{ij}\mathrm{d}x_{kl} + a_2 \underline{\hat{R}}^{-}{}^{stuv}_{ijkl} x_{st}\mathrm{d}x_{uv} + a_3 \underline{\check{R}}^{stuv}_{ijkl} x_{st}\mathrm{d}x_{uv}\\
&\quad + a_4(\underline{\hat{R}}\underline{\check{R}})^{stuv}_{ijkl} x_{st}\mathrm{d}x_{uv} + a_5\delta_{jk}\sum_s q^{-2s} x_{is}\mathrm{d}x_{sl}\\
&\quad + a_6\delta_{jk}\sum_a \grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{ia}\check{R}^{cv}_{al} x_{st}\mathrm{d}x_{uv} + a_7 \acute{R}^{ab}_{jk}\hat{R}^{-}{}^{sc}_{ia}\check{R}^{cv}_{bl}\sum_t q^{-2t} x_{st}\mathrm{d}x_{tv}\\
&\quad + a_8\sum_b \acute{R}^{st}_{ab}\acute{R}^{-}{}^{uv}_{bc}\grave{R}^{ad}_{ij}\grave{R}^{-}{}^{dc}_{kl} x_{st}\mathrm{d}x_{uv} + a_9\delta_{jk}\delta_{il}q^{-2l}\underline{\Omega}\\
&\quad + e_1\delta_{ij}q^{-2j}\sum_s q^{-2s}x_{ks}\mathrm{d}x_{sl} + e_2\delta_{ij}q^{-2j}\sum_a \grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{ka}\check{R}^{cv}_{al} x_{st}\mathrm{d}x_{uv}\\
&\quad + e_3\delta_{kl}q^{-2k}\sum_s q^{-2s}x_{is}\mathrm{d}x_{sj} + e_4\delta_{kl}q^{-2k}\sum_a \grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{ia}\check{R}^{cv}_{aj} x_{st}\mathrm{d}x_{uv}\\
&\quad + f_1\delta_{jk}\mathrm{d}x_{il} + f_2\acute{R}^{ab}_{jk}\hat{R}^{-}{}^{sc}_{ia}\check{R}^{cv}_{bl}\mathrm{d}x_{sv} + f_3\delta_{ij}q^{-2j}\mathrm{d}x_{kl}\\
&\quad + f_4\delta_{kl}q^{-2k}\mathrm{d}x_{ij} + f_5\delta_{ij}\delta_{kl}q^{-2j-2k}\underline{\Omega}\\
&\quad + b_1\sum_s q^{-2s}x_{ij}x_{ks}\mathrm{d}x_{sl} + b_2\underline{\check{R}}^{stuv}_{ijkl}\sum_w q^{-2w}x_{st}x_{uw}\mathrm{d}x_{wv}\\
&\quad + b_3\sum_a \grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{ka}\check{R}^{cv}_{al} x_{ij}x_{st}\mathrm{d}x_{uv} + b_4\underline{\hat{R}}^{xyzw}_{ijkl}\grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{za}\check{R}^{cv}_{aw} x_{xy}x_{st}\mathrm{d}x_{uv}\\
&\quad + b_5\delta_{jk}x_{ij}\underline{\Omega} + b_6\acute{R}^{ab}_{jk}\hat{R}^{-}{}^{sc}_{ia}\check{R}^{cv}_{bl}x_{sv}\underline{\Omega}\\
&\quad + g_1\delta_{ij}q^{-2j}x_{kl}\underline{\Omega} + g_2\delta_{kl}q^{-2k}x_{ij}\underline{\Omega} + c\,x_{ij}x_{kl}\underline{\Omega}.
\end{aligned}
\tag{1}
$$

In case of the "free" classification constraint, i.e. if $\mathrm{d}x_{ij}$, $(i,j) \neq (N,N)$ are supposed to be a free left module basis for $\Gamma$, all morphisms are independent. If left-module relations among the $\mathrm{d}x_{ij}$ are admitted, some summands become superfluous, resulting in an ansatz with less than 27 coefficients.

## 6. Proof of the theorem on restricted calculi

In 4.1, we mentioned the classification results on covariant first order differential $*$-calculi from [11]. We want to state first the systems of equations describing the bimodule structure of these differential calculi. Note that we use the abbreviations $\Omega^{\mathrm{S}}_{+} := \sum_{i=1}^{N} z_i \mathrm{d}z_i^*$ and $\Omega^{\mathrm{S}}_{-} := \sum_{i=1}^{N} q^{-2i} z_i^* \mathrm{d}z_i$ for the two basic invariant one-forms. In the first four families, the $\mathrm{d}z_i$ and $\mathrm{d}z_i^*$, $i = 1, \ldots, N$, form a free left-module basis for $\Gamma$.

$$
\begin{aligned}
\Gamma_{\alpha\tau} : \mathrm{d}z_k z_l &= q\alpha \hat{R}^{-\,st}_{\ kl} z_s \mathrm{d}z_t && + (q^2\alpha - 1) z_k \mathrm{d}z_l \\
&&& + q^2\alpha^2 (1 - \mathfrak{s}'_+\tau) z_k z_l \Omega^{\mathrm{S}}_{+} \\
&&& + q^2 (1 - \alpha\mathfrak{s}'_+\tau) z_k z_l \Omega^{\mathrm{S}}_{-} \\
\mathrm{d}z_k^* z_l^* &= q^{-1}\alpha^{-1} \check{R}^{st}_{kl} z_s^* \mathrm{d}z_t^* && + (q^{-2}\alpha^{-1} - 1) z_k^* \mathrm{d}z_l^* \\
&&& + (1 - \mathfrak{s}'_+\tau) z_k^* z_l^* \Omega^{\mathrm{S}}_{+} \\
&&& + \alpha^{-2} (1 - \alpha\mathfrak{s}'_+\tau) z_k^* z_l^* \Omega^{\mathrm{S}}_{-} \\
\mathrm{d}z_k z_l^* &= q^{-1}\alpha^{-1} \grave{R}^{-\,st}_{\ kl} z_s^* \mathrm{d}z_t && + (q^2\alpha - 1) z_k \mathrm{d}z_l^* \\
&&& - q^2\alpha (1 - \mathfrak{s}_+\tau) z_k z_l^* \Omega^{\mathrm{S}}_{+} - \alpha\tau q^{2k} \delta_{kl} \Omega^{\mathrm{S}}_{+} \\
&&& - \alpha^{-1} (1 - q^2\alpha\mathfrak{s}_+\tau) z_k z_l^* \Omega^{\mathrm{S}}_{-} - \tau q^{2k} \delta_{kl} \Omega^{\mathrm{S}}_{-} \\
\mathrm{d}z_k^* z_l &= q\alpha \acute{R}^{st}_{kl} z_s \mathrm{d}z_t^* && + (q^{-2}\alpha^{-1} - 1) z_k^* \mathrm{d}z_l \\
&&& - q^2\alpha (1 - \mathfrak{s}_+\tau) z_k^* z_l \Omega^{\mathrm{S}}_{+} - q^{2N} \alpha\tau \delta_{kl} \Omega^{\mathrm{S}}_{+} \\
&&& - \alpha^{-1} (1 - q^2\alpha\mathfrak{s}_+\tau) z_k^* z_l \Omega^{\mathrm{S}}_{-} - q^{2N} \tau \delta_{kl} \Omega^{\mathrm{S}}_{-}
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
\Gamma'_{\alpha\omega} : \mathrm{d}z_k z_l &= q\alpha \hat{R}^{-\,st}_{\ kl} z_s \mathrm{d}z_t && + (q^2\alpha - 1) z_k \mathrm{d}z_l \\
&&& + \omega z_k z_l \Omega^{\mathrm{S}}_{+} + (\alpha^{-1}\omega - q^2(\alpha - 1)) z_k z_l \Omega^{\mathrm{S}}_{-} \\
\mathrm{d}z_k^* z_l^* &= q^{-1}\alpha^{-1} \check{R}^{st}_{kl} z_s^* \mathrm{d}z_t^* && + (q^{-2}\alpha^{-1} - 1) z_k^* \mathrm{d}z_l^* \\
&&& + (q^2\alpha\omega^{-1} - (\alpha^{-1} - 1)) z_k^* z_l^* \Omega^{\mathrm{S}}_{+} \\
&&& + q^2\omega^{-1} z_k^* z_l^* \Omega^{\mathrm{S}}_{-} \\
\mathrm{d}z_k z_l^* &= q^{-1}\alpha^{-1} \grave{R}^{-\,st}_{\ kl} z_s^* \mathrm{d}z_t && + (q^2\alpha - 1) z_k \mathrm{d}z_l^* \\
&&& - q^2\alpha z_k z_l^* \Omega^{\mathrm{S}}_{+} - \alpha^{-1} z_k z_l^* \Omega^{\mathrm{S}}_{-} \\
\mathrm{d}z_k^* z_l &= q\alpha \acute{R}^{st}_{kl} z_s \mathrm{d}z_t^* && + (q^{-2}\alpha^{-1} - 1) z_k^* \mathrm{d}z_l \\
&&& - q^2\alpha z_k^* z_l \Omega^{\mathrm{S}}_{+} - \alpha^{-1} z_k^* z_l \Omega^{\mathrm{S}}_{-}
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
\Gamma''_{\omega\psi} : \mathrm{d}z_k z_l &= q^{-1} \hat{R}^{-\,st}_{\ kl} z_s \mathrm{d}z_t && + \omega z_k z_l \Omega^{\mathrm{S}}_{+} && + (q^2\omega\psi - 1) z_k z_l \Omega^{\mathrm{S}}_{-} \\
\mathrm{d}z_k^* z_l^* &= q \check{R}^{st}_{kl} z_s^* \mathrm{d}z_t^* && + (\psi - q^2) z_k^* z_l^* \Omega^{\mathrm{S}}_{+} && + q^2\omega^{-1} z_k^* z_l^* \Omega^{\mathrm{S}}_{-} \\
\mathrm{d}z_k z_l^* &= q \grave{R}^{-\,st}_{\ kl} z_s^* \mathrm{d}z_t && - z_k z_l^* \Omega^{\mathrm{S}}_{+} && - q^2 z_k z_l^* \Omega^{\mathrm{S}}_{-} \\
\mathrm{d}z_k^* z_l &= q^{-1} \acute{R}^{st}_{kl} z_s \mathrm{d}z_t^* && - z_k^* z_l \Omega^{\mathrm{S}}_{+} && - q^2 z_k^* z_l \Omega^{\mathrm{S}}_{-}
\end{aligned}
\tag{4}
$$

$$
(5)\quad
\begin{aligned}
\Gamma'''_{\varrho\tau}:\ \mathrm{d}z_k z_l &= q^{-1}\hat{R}^{-}{}^{st}_{kl} z_s \mathrm{d}z_t && - q^{-2}\frac{\varrho}{\tau}(\mathfrak{s}'_+\varrho - 1) z_k z_l \Omega^{\mathrm{S}}_+ - \frac{\varrho}{\tau}(\mathfrak{s}'_+\tau - q^2) z_k z_l \Omega^{\mathrm{S}}_-\\
\mathrm{d}z^*_k z^*_l &= q\check{R}^{st}_{kl} z^*_s \mathrm{d}z^*_t && - \frac{\tau}{\varrho}(\mathfrak{s}'_+\varrho - 1) z^*_k z^*_l \Omega^{\mathrm{S}}_+ - q^2\frac{\tau}{\varrho}(\mathfrak{s}'_+\tau - q^2) z^*_k z^*_l \Omega^{\mathrm{S}}_-\\
\mathrm{d}z_k z^*_l &= q\grave{R}^{-}{}^{st}_{kl} z^*_s \mathrm{d}z_t && - q^{-2}\varrho q^{2k}\delta_{kl}\Omega^{\mathrm{S}}_+ - \tau q^{2k}\delta_{kl}\Omega^{\mathrm{S}}_-\\
& && + (\mathfrak{s}_+\varrho - 1) z_k z^*_l \Omega^{\mathrm{S}}_+ + q^2(\mathfrak{s}_+\tau - 1) z_k z^*_l \Omega^{\mathrm{S}}_-\\
\mathrm{d}z^*_k z_l &= q^{-1}\acute{R}^{st}_{kl} z_s \mathrm{d}z^*_t && - q^{2N-2}\varrho\delta_{kl}\Omega^{\mathrm{S}}_+ - q^{2N}\tau\delta_{kl}\Omega^{\mathrm{S}}_-\\
& && + (\mathfrak{s}_+\varrho - 1) z^*_k z_l \Omega^{\mathrm{S}}_+ + q^2(\mathfrak{s}_+\tau - 1) z^*_k z_l \Omega^{\mathrm{S}}_-
\end{aligned}
$$

In the following three families of calculi, $\mathrm{d}z_i$, $\mathrm{d}z^*_i$ still generate $\Gamma$ as a left module but no longer as a free one. Instead, all left-module relations are algebraically generated by $\Omega^{\mathrm{S}}_+ + \lambda\Omega^{\mathrm{S}}_- = 0$ where $\lambda$ is the (real) parameter of the families of calculi.

$$
(6)\quad
\begin{aligned}
\tilde{\Gamma}_\lambda:\ \mathrm{d}z_k z_l &= q\lambda^{-1}\hat{R}^{-}{}^{st}_{kl} z_s \mathrm{d}z_t && + (q^2\lambda^{-1} - 1) z_k \mathrm{d}z_l && + q^2\lambda^{-1}(\lambda^{-1} - 1) z_k z_l \Omega^{\mathrm{S}}_+\\
\mathrm{d}z^*_k z^*_l &= q^{-1}\lambda\check{R}^{st}_{kl} z^*_s \mathrm{d}z^*_t && + (q^{-2}\lambda - 1) z^*_k \mathrm{d}z^*_l && - (\lambda - 1) z^*_k z^*_l \Omega^{\mathrm{S}}_+\\
\mathrm{d}z_k z^*_l &= q^{-1}\lambda\grave{R}^{-}{}^{st}_{kl} z^*_s \mathrm{d}z_t && + (q^2\lambda^{-1} - 1) z_k \mathrm{d}z^*_l && - (q^2\lambda^{-1} - 1) z_k z^*_l \Omega^{\mathrm{S}}_+\\
\mathrm{d}z^*_k z_l &= q\lambda^{-1}\acute{R}^{st}_{kl} z_s \mathrm{d}z^*_t && + (q^{-2}\lambda - 1) z^*_k \mathrm{d}z_l && - (q^2\lambda^{-1} - 1) z^*_k z_l \Omega^{\mathrm{S}}_+
\end{aligned}
$$

$$
(7)\quad
\begin{aligned}
\tilde{\Gamma}'_\lambda:\ \mathrm{d}z_k z_l &= q^{-1}\hat{R}^{-}{}^{st}_{kl} z_s \mathrm{d}z_t && - \lambda^{-1}(q^4\lambda^{-1} - 1) z_k z_l \Omega^{\mathrm{S}}_+\\
\mathrm{d}z^*_k z^*_l &= q\check{R}^{st}_{kl} z^*_s \mathrm{d}z^*_t && - q^{-2}\lambda(q^4\lambda^{-1} - 1) z^*_k z^*_l \Omega^{\mathrm{S}}_+\\
\mathrm{d}z_k z^*_l &= q\grave{R}^{-}{}^{st}_{kl} z^*_s \mathrm{d}z_t && + (q^2\lambda^{-1} - 1) z_k z^*_l \Omega^{\mathrm{S}}_+\\
\mathrm{d}z^*_k z_l &= q^{-1}\acute{R}^{st}_{kl} z_s \mathrm{d}z^*_t && + (q^2\lambda^{-1} - 1) z^*_k z_l \Omega^{\mathrm{S}}_+
\end{aligned}
$$

$\tilde{\Gamma}''_\lambda, \lambda \notin \{0, \infty\}:$

$$
(8)\quad
\begin{aligned}
\mathrm{d}z_k z_l &= q^{-1}\hat{R}^{-}{}^{st}_{kl} z_s \mathrm{d}z_t\\
\mathrm{d}z^*_k z^*_l &= q\check{R}^{st}_{kl} z^*_s \mathrm{d}z^*_t\\
\mathrm{d}z_k z^*_l &= q\grave{R}^{-}{}^{st}_{kl} z^*_s \mathrm{d}z_t && + q^{-2}\mathfrak{s}'_+{}^{-1}(q^4\lambda^{-1} - 1) q^{2k}\delta_{kl}\Omega^{\mathrm{S}}_+\\
& && - \mathfrak{s}'_+{}^{-1}(q^{2N+2}\lambda^{-1} - 1) z_k z^*_l \Omega^{\mathrm{S}}_+\\
\mathrm{d}z^*_k z_l &= q^{-1}\acute{R}^{st}_{kl} z_s \mathrm{d}z^*_t && + q^{2N-2}\mathfrak{s}'_+{}^{-1}(q^4\lambda^{-1} - 1)\delta_{kl}\Omega^{\mathrm{S}}_+\\
& && - \mathfrak{s}'_+{}^{-1}(q^{2N+2}\lambda^{-1} - 1) z^*_k z_l \Omega^{\mathrm{S}}_+
\end{aligned}
$$

$$
(9)\quad
\begin{aligned}
\tilde{\Gamma}''_0:\ \mathrm{d}z_k z_l &= q^{-1}\hat{R}^{-}{}^{st}_{kl} z_s \mathrm{d}z_t\\
\mathrm{d}z^*_k z^*_l &= q\check{R}^{st}_{kl} z^*_s \mathrm{d}z^*_t\\
\mathrm{d}z_k z^*_l &= q\grave{R}^{-}{}^{st}_{kl} z^*_s \mathrm{d}z_t - q^{-2N+2}\mathfrak{s}'_+{}^{-1} q^{2k}\delta_{kl}\Omega^{\mathrm{S}}_- + q^2\mathfrak{s}'_+{}^{-1} z_k z^*_l \Omega^{\mathrm{S}}_-\\
\mathrm{d}z^*_k z_l &= q^{-1}\acute{R}^{st}_{kl} z_s \mathrm{d}z^*_t - q^2\mathfrak{s}'_+{}^{-1}\delta_{kl}\Omega^{\mathrm{S}}_- + q^2\mathfrak{s}'_+{}^{-1} z^*_k z_l \Omega^{\mathrm{S}}_-
\end{aligned}
$$

$$
(10)\quad
\begin{aligned}
\tilde{\Gamma}''_\infty:\ \mathrm{d}z_k z_l &= q^{-1}\hat{R}^{-}{}^{st}_{kl} z_s \mathrm{d}z_t\\
\mathrm{d}z^*_k z^*_l &= q\check{R}^{st}_{kl} z^*_s \mathrm{d}z^*_t\\
\mathrm{d}z_k z^*_l &= q\grave{R}^{-}{}^{st}_{kl} z^*_s \mathrm{d}z_t - q^{-2}\mathfrak{s}'_+{}^{-1} q^{2k}\delta_{kl}\Omega^{\mathrm{S}}_+ + \mathfrak{s}'_+{}^{-1} z_k z^*_l \Omega^{\mathrm{S}}_+\\
\mathrm{d}z^*_k z_l &= q^{-1}\acute{R}^{st}_{kl} z_s \mathrm{d}z^*_t - q^{2N-2}\mathfrak{s}'_+{}^{-1}\delta_{kl}\Omega^{\mathrm{S}}_+ + \mathfrak{s}'_+{}^{-1} z^*_k z_l \Omega^{\mathrm{S}}_+
\end{aligned}
$$

To compute restrictions of these calculi to the sub-algebra $\mathrm{CP}_q^{N-1}$, we exploit the results of the preceding section. We rewrite $\mathrm{d}x_{ij}\cdot x_{kl}$ and all the 27 summands occurring on the right-hand side of (1) in terms of the generators of $\mathrm{S}_q^{2N-1}$ via the embedding $x_{ij}=z_iz_j^*$ and Leibniz rule. After transforming all these terms to left-module expressions by applying the bimodule structure of a given differential calculus over $\mathrm{S}_q^{2N-1}$, the ansatz equation (1) is written in left-module expressions from the $\mathrm{S}_q^{2N-1}$ calculus. By comparing coefficients (carefully observing given relations), it is easily determined whether equation (1) can be satisfied for appropriate values of the coefficients $a_1$, $a_2$, … , $c$.—Moreover, since relations in a calculus must cancel out invariant subspaces for certain co-representations, one can find out which relations occur in a restricted calculus also by rewriting expressions for morphisms in terms of $\mathrm{S}_q^{2N-1}$ and checking whether some of them become zero.

We demonstrate the procedure for $\Gamma_{\alpha\tau}$ with the bimodule structure (2). For abbreviation, let $\Omega_\alpha^{\mathrm{S}}:=\alpha\Omega_+^{\mathrm{S}}+\Omega_-^{\mathrm{S}}$. From the equalities

$$\begin{aligned}\mathrm{d}x_{ij}&=q^{-1}\alpha^{-1}\grave{R}^{-}{}^{st}_{ij}z_s^*\mathrm{d}z_t+q^2\alpha z_i\mathrm{d}z_j^*-q^2\alpha(1-\mathfrak{s}_+\tau)z_iz_j^*\Omega_+^{\mathrm{S}}\\&\quad-\alpha\tau\delta_{ij}q^{2i}\Omega_+^{\mathrm{S}}-\alpha^{-1}(1-q^2\alpha\mathfrak{s}_+\tau)z_iz_j^*\Omega_-^{\mathrm{S}}-\tau\delta_{ij}q^{2j}\Omega_-^{\mathrm{S}}\\ \sum_{k=1}^N q^{-2k}x_{ik}\mathrm{d}x_{kj}&=\alpha z_i\mathrm{d}z_j^*-\alpha(1-\mathfrak{s}'_+\tau)z_iz_j^*\Omega_+^{\mathrm{S}}+\mathfrak{s}'_+\tau z_iz_j^*\Omega_-^{\mathrm{S}}\\ \sum_{a=1}^N \grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{ia}\check{R}^{cv}_{aj}x_{st}\mathrm{d}x_{uv}&\\ &=\alpha^{-1}\grave{R}^{-}{}^{st}_{ij}z_s^*\mathrm{d}z_t+q\alpha\mathfrak{s}'_+\tau z_iz_j^*\Omega_+^{\mathrm{S}}-q\alpha^{-1}(1-\alpha\mathfrak{s}'_+\tau)z_iz_j^*\Omega_-^{\mathrm{S}}\\ \underline{\Omega}&=\mathfrak{s}'_+\tau\Omega_\alpha^{\mathrm{S}}\end{aligned}$$

it can be seen that the relation

$$\begin{aligned}\mathrm{d}x_{ij}&=q^2\sum_{k=1}^N q^{-2k}x_{ik}\mathrm{d}x_{kj}+q^{-1}\sum_{a=1}^N\grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{ia}\check{R}^{cv}_{aj}x_{st}\mathrm{d}x_{uv}\\&\quad-\frac{\mathfrak{s}''_+}{\mathfrak{s}'_+}x_{ij}\underline{\Omega}-\frac{1}{\mathfrak{s}'_+}\delta_{ij}q^{2j}\underline{\Omega}\end{aligned}\tag{11}$$

holds. This implies immediately that the morphisms corresponding with the coefficients $a_1$, $a_2$, $a_3$, $a_4$, $f_1$, $f_2$, $f_3$, $f_4$ in equation (1) are linearly dependent on the other ones and can be omitted. We therefore continue by rewriting expressions from the right-hand side of (1)—i.e. the summands that can occur in the bimodule structure of a covariant differential calculus over $\mathrm{CP}_q^{N-1}$—in terms of the generators of $\mathrm{S}_q^{2N-1}$.

$$\begin{aligned}\mathrm{d}x_{ij}\cdot x_{kl}&=q^{-1}\alpha^{-1}\grave{R}^{-}{}^{uv}_{bl}\hat{R}^{-}{}^{tb}_{ak}\grave{R}^{-}{}^{sa}_{ij}z_s^*z_tz_u^*\mathrm{d}z_v\\&\quad+q^2\alpha\check{R}^{uv}_{al}\acute{R}^{ta}_{jk}z_iz_tz_u^*\mathrm{d}z_v^*+(q^4-q^{-2}\alpha^{-1})z_iz_j^*z_kz_l^*\Omega_-^{\mathrm{S}}\\&\quad-q^{-1}\tau\delta_{ab}q^{2a}\hat{R}^{-}{}^{sa}_{ic}\check{R}^{bt}_{dl}\acute{R}^{cd}_{jk}z_sz_t^*\Omega_\alpha^{\mathrm{S}}-q^{2N+2}\tau\delta_{jk}z_iz_l^*\Omega_\alpha^{\mathrm{S}}\\&\quad-q^{-2}\tau\delta_{ij}q^{2j}z_kz_l^*\Omega_\alpha^{\mathrm{S}}+(-q^4+\tau+q^2\tau+q^4\mathfrak{s}_+\tau)z_iz_j^*z_kz_l^*\Omega_\alpha^{\mathrm{S}}\end{aligned}$$

$$\underline{\hat{R}}{}^{xyzw}_{ijkl}\grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{za}\check{R}^{cv}_{aw}x_{xy}x_{st}\mathrm{d}x_{uv} = q\alpha^{-1}\grave{R}^{-}{}^{uv}_{bl}\hat{R}^{-}{}^{tb}_{ak}\grave{R}^{-}{}^{sa}_{ij}z^*_s z_t z^*_u \mathrm{d}z_v$$
$$+ \alpha\mathfrak{s}'_+\tau z_i z^*_j z_k z^*_l \Omega^{\mathrm{S}}_+ - \alpha^{-1}(1-\alpha\mathfrak{s}'_+\tau) z_i z^*_j z_k z^*_l \Omega^{\mathrm{S}}_-$$
$$\underline{\check{R}}{}^{stuv}_{ijkl}\sum_w q^{-2w}x_{st}x_{uw}\mathrm{d}x_{wv} = q^{-1}\alpha\check{R}^{uv}_{al}\acute{R}^{ta}_{jk}z_i z_t z^*_u \mathrm{d}z^*_v$$
$$- q\alpha(1-\mathfrak{s}'_+\tau) z_i z^*_j z_k z^*_l \Omega^{\mathrm{S}}_+ + q\mathfrak{s}'_+\tau z_i z^*_j z_k z^*_l \Omega^{\mathrm{S}}_-$$

which, together with the preceding equations, imply

$$\begin{aligned}\mathrm{d}x_{ij}\cdot x_{kl} &- q^{-2}\underline{\hat{R}}{}^{xyzw}_{ijkl}\grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{za}\check{R}^{cv}_{aw}x_{xy}x_{st}\mathrm{d}x_{uv}\\ &- q^{3}\underline{\check{R}}{}^{stuv}_{ijkl}\sum_w q^{-2w}x_{st}x_{uw}\mathrm{d}x_{wv}\\ &= -\mathfrak{s}'_+{}^{-1}\big(q^{-2}\mathfrak{s}^{\mathsf{IV}}_+ x_{ij}x_{kl} + q^{-1}\delta_{ab}q^{2a}\hat{R}^{-}{}^{sa}_{ic}\check{R}^{bt}_{dl}\acute{R}^{cd}_{jk}x_{st}\\ &\qquad + q^{2N+2}\delta_{jk}x_{il} + q^{-2}\delta_{ij}q^{2j}x_{kl}\big)\underline{\Omega}\end{aligned} \tag{12}$$

and therefore the bimodule structure of $\tilde{\underline{\Gamma}}$ stated in Theorem 1. By rewriting all the morphisms from the right-hand side of (1), it is easily seen that they display no more linear dependencies than those implied by the relation (11). From these considerations, it is clear that $\tilde{\underline{\Gamma}}$ is the restriction of $\Gamma_{\alpha\tau}$ to the subalgebra $\mathrm{CP}^{N-1}_q$.

Doing completely analogous calculations one proves the other restriction statements of Theorem 1. □

## 7. Proof of the classification theorem

Our proof of Theorem 2 is organised in four parts. The first part deals with the possible coefficients in the classification constraints of the second and third case while the other three ones are devoted each to one of the constraint settings, showing that in each case there exists one uniquely determined calculus.

7.1. **Determination of possible coefficients of the constraint relations.** First we show that if all left-module relations in a covariant first order differential calculus are to be generated by one family of relations of type

$$\mathrm{d}x_{ij} = A\sum_{s=1}^{N} q^{-2s}x_{is}\mathrm{d}x_{sj} + B\sum_{a=1}^{N}\grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{ia}\check{R}^{cv}_{aj}x_{st}\mathrm{d}x_{uv} + Cx_{ij}\underline{\Omega} + D\delta_{ij}q^{2j}\underline{\Omega}$$

then the coefficients $A$, $B$, $C$, and $D$ have to take the values which hold for $\tilde{\underline{\Gamma}}$. In fact, substituting $\mathrm{d}x_{jk}$ in $\sum_{j=1}^{N} q^{-2j}x_{ij}\mathrm{d}x_{jk}$ by the above relation yields

$$\begin{aligned}\sum_{j=1}^{N} q^{-2j}x_{ij}\mathrm{d}x_{jk} &= A\sum_{j=1}^{N} q^{-2j}\sum_{s=1}^{N} q^{-2s}x_{ij}x_{js}\mathrm{d}x_{sk}\\ &\quad + B\sum_{j=1}^{N} q^{-2j}\sum_{a=1}^{N}\grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{ja}\check{R}^{cv}_{ak}x_{ij}x_{st}\mathrm{d}x_{uv}\\ &\quad + C\sum_{j=1}^{N} q^{-2j}x_{ij}x_{jk}\underline{\Omega} + D\sum_{j=1}^{N} q^{-2j}\delta_{jk}q^{2k}\underline{\Omega}\end{aligned}$$

$$= q^{-2}A\sum_{j=1}^{N} q^{-2j}x_{ij}\mathrm{d}x_{jk} + (q^{-1}B + q^{-2}C + D)x_{ik}\underline{\Omega}$$

which implies by coefficient comparison

$$A = q^2; \qquad\qquad q^{-1}B + q^{-2}C + D = 0.$$

An analogous substitution for $\mathrm{d}x_{uv}$ in $\sum_{a=1}^{N} \grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{ia}\check{R}^{cv}_{aj}x_{st}\mathrm{d}x_{uv}$ leads to

$$\begin{aligned}\sum_{a=1}^{N} \grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{ia}\check{R}^{cv}_{aj}x_{st}\mathrm{d}x_{uv} &= qB\sum_{a=1}^{N} \grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{ia}\check{R}^{cv}_{aj}x_{st}\mathrm{d}x_{uv}\\ &\quad + \big(q^{-1}A - (q^2-1)B + q^{-1}C + qD\big)x_{ij}\underline{\Omega}\end{aligned}$$

and therefore

$$B = q^{-1}; \qquad\qquad q^{-1}A - (q^2-1)B + q^{-1}C + qD = 0.$$

By inserting the values of $A$ and $B$, the last equation simplifies to

$$q^{-2}C + D = q^{-2}.$$

Finally, inserting the assumed relation in $\sum_{i=1}^{N} \mathrm{d}x_{ii} = 0$ gives

$$\begin{aligned}0 &= A\sum_{i=1}^{N}\sum_{j=1}^{N} q^{-2j}x_{ij}\mathrm{d}x_{ji} + B\sum_{i=1}^{N}\sum_{a=1}^{N} \grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{ia}\check{R}^{cv}_{ai}x_{st}\mathrm{d}x_{uv}\\ &\quad + C\sum_{i=1}^{N} x_{ii}\underline{\Omega} + D\sum_{i=1}^{N} q^{2i}\underline{\Omega}\\ &= (A + qB + C + q^2\mathfrak{s}_+D)\underline{\Omega}\end{aligned}$$

and thus

$$0 = A + qB + C + q^2\mathfrak{s}_+D = q^2 + q^2\mathfrak{s}'_+D$$

which implies

$$D = -\frac{1}{\mathfrak{s}'_+}; \qquad\qquad C = -\frac{\mathfrak{s}''_+}{\mathfrak{s}'_+}.$$

If all left-module relations in a covariant first order differential calculus are assumed to be generated by the relations

$$\begin{aligned}\mathrm{d}x_{ij} &= A\sum_{s=1}^{N} q^{-2s}x_{is}\mathrm{d}x_{sj} + B\sum_{a=1}^{N} \grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{ia}\check{R}^{cv}_{aj}x_{st}\mathrm{d}x_{uv}\\ \underline{\Omega} &= 0,\end{aligned}$$

similar (just easier) calculations as above lead directly to

$$A = q^2; \qquad\qquad B = q^{-1}.$$

Now we can prove the classification results for the different constraint settings under consideration. To accomplish this, we use the respective bimodule

structure ansatzes and relations to evaluate necessary conditions which result from the definition of differential calculus and the algebra structure of $\mathrm{CP}_q^{N-1}$. Since it appears hopeless to do by hand the extensive calculations involved (perhaps except for the case of the most reduced classification constraint), a special-purpose computer algebra program written by the author was employed to carry out the substitutions and term-reductions with R-matrices. It should be emphasised that only substitutions of R-matrix expressions via given relations were done automatically, a detailed discussion of the reduction strategy thus being not necessary. The linear independence of the summands in the resulting expressions was checked manually. Because coefficients have to be compared for expressions containing R-matrices and $x_{ij}$, $\mathrm{d}x_{ij}$ with up to 6 free indices, linear independence is clear in some cases only for $N \geq 6$; that's why this assumption is made in the uniqueness statements of the theorem.

7.2. **Case (iii).** By the assumed relations, the general ansatz (1) is reduced to 12 summands:

$$\begin{aligned}
\mathrm{d}x_{ij}x_{kl} = {} & a_5\delta_{jk}\sum_s q^{-2s}x_{is}\mathrm{d}x_{sl} + a_6\delta_{jk}\sum_a \grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{ia}\check{R}^{cv}_{al}x_{st}\mathrm{d}x_{uv} \\
& + a_7\acute{R}^{ab}_{jk}\hat{R}^{-}{}^{sc}_{ia}\check{R}^{cv}_{bl}\sum_t q^{-2t}x_{st}\mathrm{d}x_{tv} + a_8\sum_b \acute{R}^{st}_{ab}\acute{R}^{-}{}^{uv}_{bc}\grave{R}^{ad}_{ij}\grave{R}^{-}{}^{dc}_{kl}x_{st}\mathrm{d}x_{uv} \\
& + e_1\delta_{ij}q^{-2j}\sum_s q^{-2s}x_{ks}\mathrm{d}x_{sl} + e_2\delta_{ij}q^{-2j}\sum_a \grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{ka}\check{R}^{cv}_{al}x_{st}\mathrm{d}x_{uv} \\
& + e_3\delta_{kl}q^{-2k}\sum_s q^{-2s}x_{is}\mathrm{d}x_{sj} + e_4\delta_{kl}q^{-2k}\sum_a \grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{ia}\check{R}^{cv}_{aj}x_{st}\mathrm{d}x_{uv} \\
& + b_1\sum_s q^{-2s}x_{ij}x_{ks}\mathrm{d}x_{sl} + b_2\check{\underline{R}}^{stuv}_{ijkl}\sum_w q^{-2w}x_{st}x_{uw}\mathrm{d}x_{wv} \\
& + b_3\sum_a \grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{ka}\check{R}^{cv}_{al}x_{ij}x_{st}\mathrm{d}x_{uv} + b_4\hat{\underline{R}}^{xyzw}_{ijkl}\grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{za}\check{R}^{cv}_{aw}x_{xy}x_{st}\mathrm{d}x_{uv}.
\end{aligned} \tag{13}$$

We use the following necessary conditions for first order differential calculi on $\mathrm{CP}_q^{N-1}$ $(1 \leq i,j,k,l,m \leq N)$:

$$\left(\sum_{i=1}^{N}\mathrm{d}x_{ii}\right)x_{jk} = 0 \tag{14}$$

$$\mathrm{d}x_{ij}\sum_{k=1}^{N}x_{kk} - \mathrm{d}x_{ij} = 0 \tag{15}$$

$$\sum_{j=1}^{N}q^{-2j}x_{ij}\mathrm{d}x_{jk} - q^{-2}\mathrm{d}x_{ik} + \sum_{j=1}^{N}q^{-2j}\mathrm{d}x_{ij}\cdot x_{jk} = 0 \tag{16}$$

$$\mathrm{d}x_{ij}\cdot x_{kl} + x_{ij}\mathrm{d}x_{kl} - q\check{\underline{R}}^{-}{}^{stuv}_{ijkl}(\mathrm{d}x_{st}\cdot x_{uv} + x_{st}\mathrm{d}x_{uv}) = 0 \tag{17}$$

$$\mathrm{d}x_{ij}\cdot x_{kl} + x_{ij}\mathrm{d}x_{kl} - q^{-1}\hat{\underline{R}}^{stuv}_{ijkl}(\mathrm{d}x_{st}\cdot x_{uv} + x_{st}\mathrm{d}x_{uv}) = 0 \tag{18}$$

$$\sum_{m=1}^{N}q^{-2m}\mathrm{d}x_{ij}\cdot x_{km}x_{ml} - q^{-2}\mathrm{d}x_{ij}\cdot x_{kl} = 0 \tag{19}$$

(the first two of which, as the last one, follow immediately from $\sum\limits_{i=1}^{N} x_{ii} = 1$ while the remaining ones are the result of deriving algebra relations via the Leibniz rule). Because of (13), we obtain from condition (14)

$$\begin{aligned}0 &= (a_5 + q^{2N+1}a_7 + q^2\mathfrak{s}_+e_1 + b_1)\sum_{m=1}^{N} q^{-2m}x_{jm}\mathrm{d}x_{mk}\\ &\quad + (a_6 + q^{2N+1}a_8 + q^2\mathfrak{s}_+e_2 + b_3)\sum_{a=1}^{N} \grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{ja}\check{R}^{cv}_{ak}x_{st}\mathrm{d}x_{uv}.\end{aligned}$$

The two R-matrix expressions on the right-hand side are linearly independent such that we can compare coefficients to obtain two equations that must hold for the coefficients of (13). The other conditions are evaluated similarly.

By this method, the first five conditions provide us with a (redundant) set of equations for the coefficients of (13),

$$\begin{gathered}b_1 + a_5 + q^2\mathfrak{s}_+e_1 + q^{2N+1}a_7 = 0; \quad qb_3 + qa_6 + q^3\mathfrak{s}_+e_2 + q^{2N+2}a_8 = 0;\\ q^{-1}b_2 + qa_7 + a_5 + q^2\mathfrak{s}_+e_3 = q^2; \qquad q^2b_4 + q^2a_8 + qa_6 + q^3\mathfrak{s}_+e_4 = 1;\\ q^{-2}b_1 + e_3 + e_1 + q^{-2N}\mathfrak{s}_+a_5 = 0; \qquad b_4 + qe_4 + qe_2 + q^{-2N+1}\mathfrak{s}_+a_6 = q^{-2};\\ q^{-1}b_2 = b_1 + q^2; \qquad e_1 = q^{-1}a_7; \qquad q^2e_3 = q^{-2N}a_5; \qquad qe_2 = a_8;\\ q^3e_4 = q^{-2N+1}a_6; \quad b_4 = q^{-1}b_3 + q^{-2}; \quad q^{-2}e_3 = q^{-1}a_7; \quad q^{-1}e_4 = a_8;\\ e_1 + (q - q^{-1})a_7 - q^{-2N-2}a_5 = 0; \quad qe_2 + (q^2 - 1)a_8 - q^{-2N-1}a_6 = 0;\\ a_5 - q^{2N}e_1 - q^{2N-1}(q^2 - 1)a_7 - (1 - q^{-2})a_5 = 0;\\ qa_6 - q^{2N-1}e_2 - q^{2N}(q^2 - 1)a_8 - (q - q^{-1})a_6 = 0,\end{gathered}$$

which is fulfilled if the coefficients depend on two complex parameters $\alpha$, $\beta$ via

$$\begin{aligned}&a_5 = \alpha; && a_6 = \beta; && a_7 = q^{-2N-3}\alpha; && a_8 = q^{-2N-3}\beta;\\ &e_1 = q^{-2N-4}\alpha; && e_2 = q^{-2N-4}\beta; && e_3 = q^{-2N-2}\alpha; && e_4 = q^{-2N-2}\beta;\\ &b_1 = -(1 + q^{-2} + q^{-2N-2}\mathfrak{s}_+)\alpha; &&&& b_2 = q^3 - (q + q^{-1} + q^{-2N-1}\mathfrak{s}_+)\alpha;\\ &b_3 = -(1 + q^{-2} + q^{-2N-2}\mathfrak{s}_+)\beta; &&&& b_4 = q^{-2} - (q^{-1} + q^{-3} + q^{-2N-3}\mathfrak{s}_+)\beta.\end{aligned}$$

By evaluating condition (19) using this two-parameter form of all coefficients and doing coefficient comparison, we obtain finally $\alpha = \beta = 0$ which makes clear that $\tilde{\tilde{\underline{\Gamma}}}$ is the only covariant first order differential calculus under this constraint setting.

7.3. **Case (ii).** By the relations of this case, the bimodule structure ansatz (1) is reduced to 19 members, namely

$$
\begin{aligned}
\mathrm{d}x_{ij}x_{kl} &= a_5\delta_{jk}\sum_s q^{-2s}x_{is}\mathrm{d}x_{sl} + a_6\delta_{jk}\sum_a \grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{ia}\check{R}^{cv}_{al}x_{st}\mathrm{d}x_{uv} \\
&\quad + a_7\acute{R}^{ab}_{jk}\hat{R}^{-}{}^{sc}_{ia}\check{R}^{cv}_{bl}\sum_t q^{-2t}x_{st}\mathrm{d}x_{tv} + a_8\sum_b \acute{R}^{st}_{ab}\acute{R}^{-}{}^{uv}_{bc}\grave{R}^{ad}_{ij}\grave{R}^{-}{}^{dc}_{kl}x_{st}\mathrm{d}x_{uv} \\
&\quad + a_9\delta_{jk}\delta_{il}q^{-2l}\underline{\Omega} + f_5\delta_{ij}\delta_{kl}q^{-2j-2k}\underline{\Omega} \\
&\quad + e_1\delta_{ij}q^{-2j}\sum_s q^{-2s}x_{ks}\mathrm{d}x_{sl} + e_2\delta_{ij}q^{-2j}\sum_a \grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{ka}\check{R}^{cv}_{al}x_{st}\mathrm{d}x_{uv} \\
&\quad + e_3\delta_{kl}q^{-2k}\sum_s q^{-2s}x_{is}\mathrm{d}x_{sj} + e_4\delta_{kl}q^{-2k}\sum_a \grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{ia}\check{R}^{cv}_{aj}x_{st}\mathrm{d}x_{uv} \\
&\quad + b_1\sum_s q^{-2s}x_{ij}x_{ks}\mathrm{d}x_{sl} + b_2\underline{\check{R}}^{stuv}_{ijkl}\sum_w q^{-2w}x_{st}x_{uw}\mathrm{d}x_{wv} \\
&\quad + b_3\sum_a \grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{ka}\check{R}^{cv}_{al}x_{ij}x_{st}\mathrm{d}x_{uv} + b_4\underline{\hat{R}}^{xyzw}_{ijkl}\grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{za}\check{R}^{cv}_{aw}x_{xy}x_{st}\mathrm{d}x_{uv} \\
&\quad + b_5\delta_{jk}x_{ij}\underline{\Omega} + b_6\acute{R}^{ab}_{jk}\hat{R}^{-}{}^{sc}_{ia}\check{R}^{cv}_{bl}x_{sv}\underline{\Omega} \\
&\quad + g_1\delta_{ij}q^{-2j}x_{kl}\underline{\Omega} + g_2\delta_{kl}q^{-2k}x_{ij}\underline{\Omega} + cx_{ij}x_{kl}\underline{\Omega}.
\end{aligned} \tag{20}
$$

Now we evaluate again the conditions (14)–(18). By carrying out the coefficient comparison completely (for all morphisms appearing) for (17) and (18) but only in part for the first three conditions (since some of the expressions are very lengthy), we obtain the equations

$$
\begin{gathered}
b_1 + a_5 + q^2\mathfrak{s}_+e_1 + q^{2N+1}a_7 = 0; \qquad qb_3 + qa_6 + q^3\mathfrak{s}_+e_2 + q^{2N+2}a_8 = 0; \\
q^{-1}b_2 + qa_7 + a_5 + q^2\mathfrak{s}_+e_3 = q^2; \qquad q^2b_4 + q^2a_8 + qa_6 + q^3\mathfrak{s}_+e_4 = 1; \\
q^{-2}b_1 + e_3 + e_1 - \mathfrak{s}_+a_5 = 0; \qquad b_4 + qe_4 + qe_2 - q\mathfrak{s}_+a_6 = q^{-2}; \\
q^{-1}b_2 = b_1 + q^2; \qquad e_1 = q^{-1}a_7; \qquad q^2e_3 = q^{-2N}a_5; \\
qe_2 = a_8; \qquad q^3e_4 = q^{-2N+1}a_6; \qquad g_1 = q^{-1}b_6; \\
b_4 = q^{-1}b_3 + q^{-2}; \quad q^{-2}e_3 = q^{-1}a_7; \qquad q^{-1}e_4 = a_8; \\
q^2g_2 = q^{-2N}b_5 + q^2{\mathfrak{s}'_+}^{-1}; \qquad e_1 = q^{-2N-2}a_5 - (q-q^{-1})a_7; \\
qe_2 = q^{-2N-1}a_6 - (q^2-1)a_8; \quad g_1 = q^{-2N-2}b_5 - (q-q^{-1})b_6; \\
q^{-2}g_2 = q^{-1}b_6 + q^{-2}{\mathfrak{s}'_+}^{-1}; \qquad a_9 = -q^{2N+2}f_5
\end{gathered}
$$

which reduces the number of independent coefficients to 7, namely $a_7$, $a_8$, $a_9$, $b_1$, $b_3$, $b_6$, and $c$, via the relations

$$
\begin{array}{llll}
a_5 = q^{2N+3}a_7; & a_6 = q^{2N+3}a_8; & b_2 = qb_1 + q^3; & b_4 = q^{-1}b_3 + q^{-2}; \\
e_1 = q^{-1}a_7; & e_2 = q^{-1}a_8; & e_3 = qa_7; & e_4 = qa_8; \\
b_5 = q^{2N+3}b_6; & f_5 = q^{-2N-2}a_9; & g_1 = q^{-1}b_6; & g_2 = qb_6 + {\mathfrak{s}'_+}^{-1}.
\end{array}
$$

Further information is obtained from the condition

(21)
$$q^2\sum_{m=1}^{N} q^{-2m}x_{im}\mathrm{d}x_{mj}\cdot x_{kl}+q^{-1}\sum_{a=1}^{N}\grave{R}^{-}{}^{tu}_{bc}\hat{R}^{-}{}^{sb}_{ia}\check{R}^{cv}_{aj}x_{st}\mathrm{d}x_{uv}\cdot x_{kl}-\mathrm{d}x_{ij}\cdot x_{kl}=0$$

which is, of course, specific to this particular constraint setting because it is derived from the imposed relation. This equation leads to $a_7=a_8=b_1=b_3=a_9=0$ leaving just two parameters $b_6$ and $c$. With these simplification, we do the remaining coefficient comparisons for condition (14) and obtain

$$g_2=0; \qquad c=-q^{-2}-q^{-4}-(q\mathfrak{s}_++q^{2N+1}+q^{2N+3})b_6,$$

therefore finally as the unique solution the coefficients of $\tilde{\underline{\Gamma}}$.

7.4. **Case (i) (free left module).** This is the most difficult case to handle. Here, the full ansatz (1) with 27 unknown coefficients applies. We start by evaluating the conditions (17) and (18). Coefficient comparison leads to the following equations for the coefficients of (1):

$$\begin{array}{llll}
a_3=qa_1+q; & a_4=qa_2; & f_3=q^{-1}f_2; & f_4=q^{-2N-2}f_1;\\
b_2=qb_1; & e_1=q^{-1}a_7; & e_3=q^{-2N-2}a_5; & e_2=q^{-1}a_8;\\
e_4=q^{-2N-2}a_6; & g_1=q^{-1}b_6; & g_2=q^{-2N-2}b_5; & f_5=q^{-2N-2}a_9;\\
a_2=q^{-1}a_1+q^{-1}; & a_4=q^{-1}a_3; & f_4=qf_2; & b_4=q^{-1}b_3;\\
e_3=qa_7; & e_4=qa_8; & g_2=qb_6; &
\end{array}$$
$$f_3=q^{-2N-2}f_1-(q-q^{-1})f_2;\quad e_1=q^{-2N-2}a_5-(q-q^{-1})a_7;$$
$$e_2=q^{-2N-2}a_6-(q-q^{-1})a_8;\quad g_1=q^{-2N-2}b_5-(q-q^{-1})b_6.$$

Using these equations, we can rewrite the coefficients of the ansatz as dependent on only 9 parameters $\alpha$, $\beta$, $\gamma$, $\delta_1$, $\delta_2$, $\delta_3$, $\varepsilon$, $\zeta$, and $c$ via

$$\begin{array}{llll}
a_1=\alpha-1; & a_2=q^{-1}\alpha; & a_3=q\alpha; & a_4=\alpha;\\
a_5=\beta; & a_6=\gamma; & a_7=q^{-2N-3}\beta; & a_8=q^{-2N-3}\gamma;\\
e_1=q^{-2N-4}\beta; & e_2=q^{-2N-4}\gamma; & e_3=q^{-2N-2}\beta; & e_4=q^{-2N-2}\gamma;\\
b_1=\delta_2; & b_2=q\delta_2; & b_3=\delta_3; & b_4=q^{-1}\delta_3;\\
b_5=\varepsilon; & b_6=q^{-2N-3}\varepsilon; & g_1=q^{-2N-4}\varepsilon; & g_2=q^{-2N-2}\varepsilon;\\
f_1=\zeta; & f_2=q^{-2N-3}\zeta; & f_3=q^{-2N-4}\zeta; & f_4=q^{-2N-2}\zeta;\\
 & f_5=q^{-2N-2}\delta_1; & a_9=\delta_1. &
\end{array} \tag{22}$$

Moreover, from (14) we obtain the conditions

$$\begin{aligned}
f_1+a_1+q^2\mathfrak{s}_+f_3+q^{2N+1}f_2&=0;\\
b_1+a_5+qa_2+q^2\mathfrak{s}_+e_1+q^{2N+1}a_7&=0;\\
b_3+a_6+a_3+q^2\mathfrak{s}_+e_2+q^{2N+1}a_8&=0;\\
c+b_5+b_4+qb_2+q^2\mathfrak{s}_+g_1+q^{2N+1}b_6&=0;
\end{aligned}$$

$$g_2 + qe_4 + e_3 + a_9 + q^2\mathfrak{s}_+ f_5 = 0;$$

which lead to further dependencies between the parameters, namely

$$\begin{aligned} \alpha &= 1 + (1 + q^{-2} + q^{-2N-2}\mathfrak{s}_+)\zeta; \\ \delta_2 &= -\alpha - (1 + q^{-2} + q^{-2N-2}\mathfrak{s}_+)\beta; \\ \delta_3 &= -q\alpha - (1 + q^{-2} + q^{-2N-2}\mathfrak{s}_+)\gamma; \\ \varepsilon &= -\beta - q\gamma - (q^2 + q^4\mathfrak{s}_+)\delta_1; \\ c &= -q^2\delta_2 - q^{-1}\delta_3 - (1 + q^{-2} + q^{-2N-2}\mathfrak{s}_+)\varepsilon, \end{aligned} \tag{23}$$

which we shall leave aside for the moment. Instead, we evaluate the condition

$$q\check{\underline{R}}^{-\,stuv}_{\ \ klmn}\mathrm{d}x_{ij} \cdot x_{st}x_{uv} - \mathrm{d}x_{ij} \cdot x_{kl}x_{mn} = 0 \tag{24}$$

with the system of coefficients reduced only by (22). This leads to

$$\varepsilon = 0; \qquad \beta = 0; \qquad \gamma = 0; \qquad \zeta = 0; \qquad \delta_1 = 0,$$

thus simplifying (23) to

$$\alpha = 1; \qquad \delta_2 = -1; \qquad \delta_3 = -q; \qquad c = q^2 + 1.$$

Backtracking the substitutions gives the coefficients of the original ansatz to be

$$\begin{gathered} a_1 = 0; \qquad a_2 = q^{-1}; \qquad a_3 = q; \qquad a_4 = 1; \\ b_1 = -1; \qquad b_2 = -q; \qquad b_3 = -q; \qquad b_4 = -1; \\ c = q^2 + 1; \\ a_5 = a_6 = a_7 = a_8 = a_9 = 0; \qquad b_5 = b_6 = g_1 = g_2 = 0; \\ e_1 = e_2 = e_3 = e_4 = 0; \qquad f_1 = f_2 = f_3 = f_4 = f_5 = 0. \end{gathered}$$

By testing the complete list of necessary conditions (again with computer-algebra reduction) one checks that all of them are fulfilled; thus the coefficient system obtained describes in fact a differential calculus, with the bimodule structure being as described in Theorem 2, case (i). This check is required only in this case since the corresponding statement for the other two cases is covered by the assertions of Theorem 1. This completes the proof of Theorem 2. □

## 8. Conclusion

The description of first order differential calculus on the quantum projective spaces forms the first step on the way to an investigation of their noncommutative geometry. Higher order differential calculus would be the next indispensable pre-requisite for formulating basic concepts of differential geometry on $\mathrm{CP}_q^{N-1}$.

At the same time, the results proved here together with previous results [6], [11] draw an outline of how covariant first order differential calculus on the quantum group $\mathrm{SU}_q(N)$ and the related quantum homogeneous spaces $\mathrm{S}_q^{2N-1}$, $\mathrm{CP}_q^{N-1}$ is linked and how different closely related quantum spaces may behave. To illustrate the latter, remember just that on $\mathrm{CP}_q^{N-1}$, differential calculi are essentially uniquely determined while on $\mathrm{S}_q^{2N-1}$ there was a vast variety of parametrical series of them.

**Acknowledgement.** The author gratefully acknowledges support of the work presented here by Deutsche Forschungsgemeinschaft within the programme of Graduiertenkolleg Quantenfeldtheorie, Leipzig.

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Martin Welk, Department of Mathematics, University of Leipzig, Augustusplatz 10, 04109 Leipzig, Germany